Preliminary Mathematics (2U)

The Quadratic Polynomial

Term 2 – Week 5

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Term 2 – Week 5 – Theory

Sketching and Solving Quadratic Inequalities:

The best way of solving quadratic inequalities is by sketching the quadratic function.

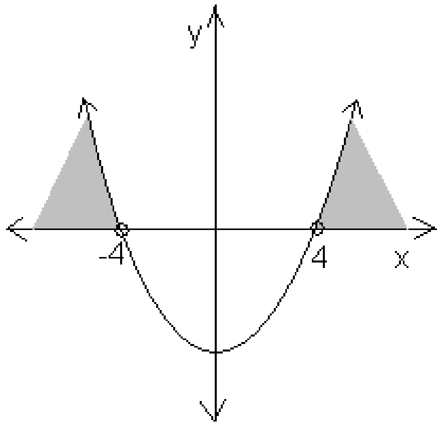
Example:

Find the values of x:

(i) $x^2 - 16 > 0$
(ii) $x^2 - 5x + 6 \leq 0$
(iii) $1 - x - 2x^2 \geq 0$
(iv) $3 + x - 2x^2 < 0$

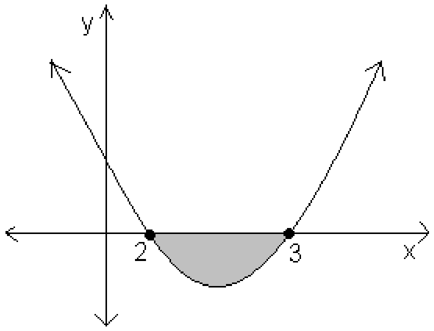
Solution:

(i) $x^2 - 16 > 0$
$(x-4)(x+4) > 0$



$\therefore x < -4, \qquad x > 4$

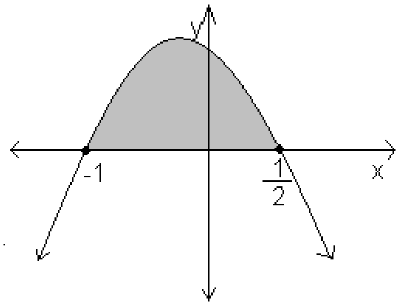
(ii) $x^2 - 5x + 6 \leq 0$
$(x-2)(x-3) \leq 0$



$\therefore 2 \leq x \leq 3$

(iii) $1 - x - 2x^2 \geq 0$

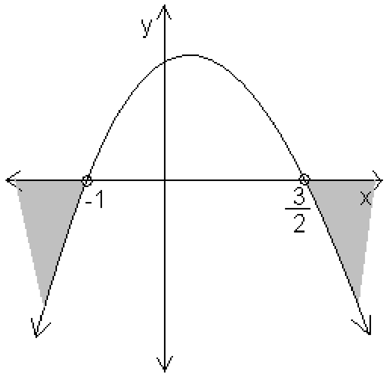
$(1-2x)(1+x) \geq 0$



$\therefore -1 \leq x \leq \frac{1}{2}$

(iv) $3 + x - 2x^2 < 0$

$(3-2x)(1+x) < 0$



$\therefore x < -1, \qquad x > \frac{3}{2}$

Equations Reducible to Quadratic Equations:

Example:

Solve the following equations:

a) $x^4 - 13x^2 + 36 = 0$
b) $4^x - 9.2^x + 8 = 0$
c) $(x-1)^4 - 5(x-1)^2 + 4 = 0$
d) $9^{x+1} - 28.3^x + 3 = 0$
e) $\sin^2 x - 3\sin x + 1 = 0$

Solution:

a) $x^4 - 13x^2 + 36 = 0$

$(x^2)^2 - 13x^2 + 36 = 0$

Let $a = x^2$

$a^2 - 13a + 36 = 0$

$(a-9)(a-4) = 0$

$a = 9, \quad a = 4$

$x^2 = 9, \quad x^2 = 4$

$\therefore x = \pm 3, \quad x = \pm 2$

b) $4^x - 9.2^x + 8 = 0$

$2^{2x} - 9.2^x + 8 = 0$

$(2^x)^2 - 9.2^x + 8 = 0$

Let $a = 2^x$

$a^2 - 9a + 8 = 0$

$(a-8)(a-1) = 0$

$a = 8, \quad a = 1$

$2^x = 8, \quad 2^x = 1$

$2^x = 2^3, \quad 2^x = 2^0$

$\therefore x = 3, \quad 0$

c) $(x-1)^4 - 5(x-1)^2 + 4 = 0$

Let $a = (x-1)^2$

$a^2 - 5a + 4 = 0$

$(a-1)(a-4) = 0$

$a = 1, \quad a = 4$

$(x-1)^2 = 1, \quad (x-1)^2 = 4$

$x - 1 = \pm 1, \quad x - 1 = \pm 2$

$\therefore x = 0, \quad 2, \quad 3, \quad -1$

d) $9^{x+1} - 28.3^x + 3 = 0$

$9.9^x - 28.3^x + 3 = 0$

$9.3^{2x} - 28.3^x + 3 = 0$

Let $a = 3^x$

$9a^2 - 28a + 3 = 0$

$(9a-1)(a-3) = 0$

$a = \frac{1}{9}, \quad a = 3$

$3^x = \frac{1}{9}, \quad 3^x = 3$

$3^x = 3^{-2}, \quad 3^x = 3^1$

$\therefore x = -2, \quad 1$

e) $2\sin^2 x - 3\sin x + 1 = 0, \quad \text{for } 0 \le x \le 360°$

Let $a = \sin x$

$$2a^2 - 3a + 1 = 0$$
$$(2a-1)(a-1) = 0$$
$$a = \frac{1}{2}, \qquad a = 1$$
$$\sin x = \frac{1}{2}, \qquad \sin x = 1$$
$$\therefore x = 30°, \qquad 150°, \qquad 90°$$

Identity of Two Quadratic Expressions:

Theorem: Two quadratic polynomials are equal iff they are equal for all values of the variable, i.e. the corresponding coefficients of like terms are equal.

If $ax^2 + bx + c \equiv Ax^2 + Bx + C$ for all values of the variable, then $a = A, b = B$ and $c = C$.

Example:

Find the values of a, b and c if $x^2 + 5x + 8 \equiv a(x+1)^2 + b(x+1) + c$.

Solution:

$$x^2 + 5x + 8 \equiv a(x+1)^2 + b(x+1) + c$$
$$\equiv a(x^2 + 2x + 1) + bx + b + c$$
$$\equiv ax^2 + 2ax + a + bx + b + c$$
$$\equiv ax^2 + (2a+b)x + a + b + c$$

Equate coefficients of corresponding like terms,

$$\therefore a = 1$$
$$2a + b = 5$$
$$2(1) + b = 5$$
$$\therefore b = 3$$
$$a + b + c = 8$$
$$1 + 5 + c = 8$$
$$\therefore c = 2$$
$$\therefore a = 1,\ b = 3,\ c = 2$$

Example:

Express $5x^2 - 6x + 4$ in the form $a(2x^2 + 1) + b(x^2 + 2x)$.

Solution:

$$5x^2 - 6x + 4 \equiv a(2x^2 + 1) + b(x^2 + 2x)$$

$$\equiv 2ax^2 + a + bx^2 + 2bx$$

$$\equiv (2a + b)x^2 + 2bx + a$$

$\therefore a = 4$

$2b = -6$

$\therefore b = -3$

$\therefore 5x^2 - 6x + 4 \equiv 4(2x^2 + 1) - 3(x^2 + 2x)$

Example:

(i) Express $x^4 - 4x^3 - 7x^2 + 22x + 24$ in the form $a(x^2 - 2x)^2 + b(x^2 - 2x) + c$.
(ii) Hence, solve $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$.

Solution:

(i)

$$x^4 - 4x^3 - 7x^2 + 22x + 24 \equiv a(x^2 - 2x)^2 + b(x^2 - 2x) + c$$

$$\equiv a(x^4 - 4x^3 + 4x^2) + bx^2 - 2bx + c$$

$$\equiv ax^4 - 4ax^3 + 4ax^2 + bx^2 - 2bx + c$$

$$\equiv ax^4 - 4ax^3 + (4a + b)x^2 - 2bx + c$$

Equate coefficient of corresponding terms,

$\therefore a = 1$

$-2b = 22$

$\therefore b = -11$

$\therefore c = 24$

$\therefore x^4 - 4x^3 - 7x^2 + 22x + 24 \equiv (x^2 - 2x)^2 - 11(x^2 - 2x) + 24$

(ii)

$$x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$$

$$(x^2 - 2x)^2 - 11(x^2 - 2x) + 24 = 0$$

Let $a = x^2 - 2x$

$$a^2 - 11a + 24 = 0$$

$$(a - 3)(a - 8) = 0$$

$$a = 3, \qquad a = 8$$

$$x^2 - 2x = 3, \qquad x^2 - 2x = 8$$

$$x^2 - 2x - 3 = 0, \qquad x^2 - 2x - 8 = 0$$

$$(x - 3)(x + 1) = 0, \qquad (x - 4)(x + 2) = 0$$

$$\therefore x = -1, \qquad 3, \qquad -2, \qquad 4$$

Example:

Find the values of a, b and c such that $x^2 + 9x + 10 \equiv (x + a)^2 + bx + c$.

Solution:

$$x^2 + 9x + 10 \equiv (x + a)^2 + bx + c$$

$$(x + a)^2 + bx + c \equiv x^2 + 9x + 10$$

$$\equiv x^2 + 9x + \left(\frac{9}{2}\right)^2 + 10 - \left(\frac{9}{2}\right)^2$$

$$\equiv \left(x + \frac{9}{2}\right)^2 - \frac{41}{4}$$

$$\therefore a = \frac{9}{2}, \qquad b = 0, \qquad c = -\frac{41}{4}$$

Term 2 – Week 5 – Homework

Sketching and Solving Quadratic Inequalities:

1. Find the values of x by sketching the quadratic function.
   a) $x^2 - 9 > 0$
   b) $x^2 - 4 \leq 0$
   c) $x^2 - 5x + 6 \geq 0$
   d) $x^2 - 2x - 15 < 0$
   e) $6x^2 + 7x - 5 > 0$
   f) $8x^2 + 5x - 3 \leq 0$
   g) $5 - 9x - 2x^2 > 0$
   h) $8 - 6x - 9x^2 \leq 0$
   i) $x^2 + 2x + 1 \geq 0$
   j) $(x - 4)^2 > 0$
   k) $x^2 \leq 0$
   l) $(x + 2)^2 < 0$
   m) $2x^2 - 3x + 4 > 0$
   n) $x(3 - x) \leq -4$
   o) $3x^2 + 2x + 1 < 0$
   p) $x^2 + x + 5 \leq 0$

Equations Reducible to Quadratics:

1. Solve the following quadratic equations.
   a) $x^4 - 17x^2 + 16 = 0$
   b) $4^x - 12.2^x + 32 = 0$
   c) $16^x - 5.4^x + 4 = 0$
   d) $2\sin^2 x - \sin x = 0, \quad 0 \leq x \leq 360°$
   e) $x^4 - 12x^2 + 27 = 0$
   f) $9^x - 12.3^x + 27 = 0$
   g) $(x^2 - 2x)^2 - 27(x^2 - 2x) + 72 = 0$
   h) $\tan^2 x - \tan x = 0, \quad 0 \leq x \leq 360°$
   i) $9x^4 - 40x^2 + 16 = 0$
   j) $(x^2 + x)^2 - 14(x^2 + x) + 24 = 0$
   k) $\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) - 10 = 0$
   l) $3^{2x+1} - 28.3^x + 9 = 0$
   m) $4^x - 17.2^x + 16 = 0$
   n) $4x^4 - 9x^2 + 2 = 0$
   o) $4\sin^2 x + 4\sin x + 1 = 0, \quad 0 \leq x \leq 360°$
   p) $2^{2x+1} - 9.2^x + 4 = 0$
   q) $\left(x - \frac{2}{x}\right)^2 - 4\left(x - \frac{2}{x}\right) - 12 = 0$

r) $15x^4 - 17x^2 + 4 = 0$
s) $(x^2 + 3x)^2 - 22(x^2 + 3x) + 72 = 0$
t) $2\cos^2 x - \cos x - 1 = 0, \quad 0 \le x \le 360°$
u) $\tan^2 x - \sqrt{3}\tan x = 0, \quad 0 \le x \le 360°$
v) $25^x - 26.5^x + 25 = 0$
w) $\sqrt{3}\tan^2 x + \left(\sqrt{3} - 1\right)\tan x - 1 = 0, \quad 0 \le x \le 360°$
x) $2\left(x + \frac{3}{x}\right)^2 - 9\left(x + \frac{3}{x}\right) + 4 = 0$

Identity of Two Quadratic Expressions:

1. Find the values of a, b and c for the following identities.
   a) $3x^2 - 2x + 4 \equiv a(x-1)^2 + b(x-1) + c$
   b) $7x^2 - x + 2 \equiv ax(x-1) + b(x^2 + x) + c$
   c) $33 + 2x - 6x^2 \equiv a(2x^2 + 7) + bx(1 - 7x) + c$
   d) $13x^2 + 20x - 7 \equiv (ax - 2)(x + 3) + bx(2 + 3x) + c(x - 1)$
   e) $5 - 2x^2 \equiv a + b(x+1) + c(x+1)^2$
   f) $x^2 - 8x + 17 \equiv a(x-2)^2 - b(x-2) + c$

2. Express $8x^2 + 10x + 4$ in the form $Ax(x+2) + B(x+1) + Cx^2$.

3. Express $10x^2 + 3x + 27$ in the form $ax^2 + bx(2x+1) + c$.

4.
   (i) Express $x^4 + 2x^3 - 7x^2 - 8x + 12$ in the form $A(x^2 + x)^2 - B(x^2 + x) + C$.
   (ii) Hence solve $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.

5.
   (i) Express $4x^4 + 4x^3 - 7x^2 - 4x + 3$ in the form $A(2x^2 + x)^2 + B(2x^2 + x) + C$.
   (ii) Hence solve $4x^4 + 4x^3 - 7x^2 - 4x + 3 = 0$.

End of homework