HSC Mathematics (2U)

Applications of Calculus to the Physical World

Term 2 – Week 3

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Term 2 – Week 3 – Theory

Exponential Growth and Decay:

In this section, we will demonstrate some physical world applications of the exponential function. We have seen earlier that if $y = e^x$, then $\frac{dy}{dx} = y = e^x$, i.e. the derivative of the exponential function is the function itself.

Natural Growth and Decay Theorem:

If $\quad y = Ae^{kt}$,

Then $\frac{dy}{dt} = kAe^{kt}$

$$\therefore \frac{dy}{dt} = ky$$

That is, the rate of change of y with respect to t is proportional to y.

Conversely, suppose the rate of change of y is proportional to y, i.e.

If $\quad \frac{dy}{dt} = ky$

Then, $y = Ae^{kt}$, where A and k are constants.

More precisely, A is the value when $t = 0$ and k is the growth rate.

Proof:

If $\quad \frac{dy}{dt} = ky$

$$\frac{dt}{dy} = \frac{1}{ky}$$

$$t = \int \frac{1}{ky}.dy$$

$$t = \frac{1}{k}\int \frac{1}{y}.dy$$

$$t = \frac{1}{k}\ln y + c$$

$$t - c = \frac{1}{k}\ln y$$

$$k(t - c) = \ln y$$

$e^{k(t-c)} = y$

$$y = e^{k(t-c)}$$

$$y = e^{kt-kc}$$

$y = e^{kt}.e^{-kc}$, but e^{-kc} is a constant so let $e^{-kc} = A$.

$$\therefore y = Ae^{kt}$$

The difference between k and $\frac{dy}{dt}$:

For $y = Ae^{kt}$ and $\frac{dy}{dt} = ky$, both k and $\frac{dy}{dt}$ are often referred to as 'growth rate', so what is the difference between them?

The value $\frac{dy}{dt}$ is the rate of growth of y per unit of time. However, since $\frac{dy}{dt} = ky$, then k is the proportional rate of growth.

Geometrically, the value k determines the steepness of the exponential curve, but $\frac{dy}{dt}$ is the gradient of the tangent to the curve.

Example:

Given $P = 20e^{-0.1t}$, find $\frac{dP}{dt}$ when

(i) $t = 2$
(ii) $P = 10$

Solution:

(i) $P = 20e^{-0.1t}$

$$\frac{dP}{dt} = -0.1 \times 20e^{-0.1t}$$

$$\frac{dP}{dt} = -2e^{-0.1t}$$

When $t = 2$,

$$\frac{dP}{dt} = -2e^{-0.1(2)}$$

$$= -2e^{-0.2}$$

$$\therefore \frac{dP}{dt} = -1.637 \qquad \text{(nearest 3 decimal places)}$$

(ii) $\frac{dP}{dt} = -0.1 \times 20e^{-0.1t}$

$\frac{dP}{dt} = -0.1P$

When $P = 10$,

$\frac{dP}{dt} = -0.1(10)$

$\therefore \frac{dP}{dt} = -1$

Example:

The rate of change of the price P of a Blu-Ray player is given by $\frac{dP}{dt} = kP$, where k is a constant and t is time in months.

(i) If initially the Blu-Ray player costs $\$950$, express P as a function of t.
(ii) If after two months the player costs $\$800$, find the value of k.
(iii) How long does it take to half the price of the player?

Solution:

(i) $\frac{dP}{dt} = kP$

$\frac{dt}{dP} = \frac{1}{kP}$

$t = \int \frac{1}{kP} . dP$

$t = \frac{1}{k} \int \frac{1}{P} . dP$

$t = \frac{1}{k} \ln P + c$

$t - c = \frac{1}{k} \ln P$

$k(t - c) = \ln P$

$e^{k(t-c)} = P$

$P = e^{k(t-c)}$

$P = e^{kt}.e^{-kc}$ $\quad$ $\left(\text{Since } e^{-kc} \text{ is a constant, so let } A = e^{-kc}\right)$

$P = Ae^{kt}$

When $t = 0, P = 950$,

$950 = Ae^0$

$\therefore A = 950$

$\therefore P = 950e^{kt}$

(ii) When $t = 2, P = 800$

$$800 = 950e^{2k}$$
$$\frac{16}{19} = e^{2k}$$
$$\ln\left(\frac{16}{19}\right) = 2k$$
$$\frac{1}{2}\ln\left(\frac{16}{19}\right) = k$$
$$\therefore k = \frac{1}{2}\ln\left(\frac{16}{19}\right)$$
$$\therefore P = 950e^{\frac{1}{2}\ln\left(\frac{16}{19}\right)t}$$

(iii) When $P = \frac{950}{2} = 475$

$$475 = 950e^{\frac{1}{2}\ln\left(\frac{16}{19}\right)t}$$
$$0.5 = e^{\frac{1}{2}\ln\left(\frac{16}{19}\right)t}$$
$$\ln 0.5 = \frac{1}{2}\ln\left(\frac{16}{19}\right)t$$
$$2\ln 0.5 = \ln\left(\frac{16}{19}\right)t$$
$$\frac{2\ln 0.5}{\ln\left(\frac{16}{19}\right)} = t$$
$$t = \frac{2\ln 0.5}{\ln\left(\frac{16}{19}\right)}$$
$$\therefore t = 8 \text{ months} \quad \text{(nearest months)}$$

$\therefore$ It takes approximately 8 months for the price of the Blu-Ray player to half.

Example:

The rate of growth of bacteria in a living cell at any time t days is given by $P = P_0e^{0.1t}$.

(i) If initially there are 10 thousands bacteria, find the number of bacteria after 10 days.
(ii) How many days does it take for the number of bacteria to triple?
(iii) Find the rate of growth when $P = 15$ thousands.

Solution:

(i) $P = P_0e^{0.1t}$

When $t = 0, P = 10$

$10 = P_0e^0$

$10 = P(1)$

$\therefore P = 10$

$P = 10e^{0.1t}$

When $t = 10$,

$P = 10e^{0.1(10)}$

$P = 10e^1$

$\therefore P = 27.18 \text{ thousands}$

(ii) When $P = 30$

$30 = 10e^{0.1t}$

$3 = e^{0.1t}$

$\ln 3 = 0.1t$

$\frac{\ln 3}{0.1} = t$

$t = \frac{\ln 3}{0.1}$

$\therefore\ t = 11 \text{ days} \quad \text{(nearest day)}$

(iii) $\frac{dP}{dt} = 0.1 \times 10e^{0.1t}$

$\frac{dP}{dt} = 0.1P$

When $P = 15$,

$\frac{dP}{dt} = 0.1 \times 15$

$\therefore \frac{dP}{dt} = 1.5 \text{ thousands per day}$

Example:

The rate of decay of a radioactive substance is proportional to the mass of the substance present at any time t. If a quarter of the substance decomposed in 100 years, what percentage of the original amount remains after 500 years?

Solution:

$M = Ae^{kt}$

$\frac{1}{4}$ of the substance is decomposed, then $\frac{3}{4}$ remains.

When $t = 100$, $M = \frac{3}{4}A$

$$\frac{3}{4}A = Ae^{100k}$$

$$\frac{3}{4} = e^{100k}$$

$$\ln\left(\frac{3}{4}\right) = 100k$$

$$\frac{1}{100}\ln\left(\frac{3}{4}\right) = k$$

$$\therefore k = \frac{1}{100}\ln\left(\frac{3}{4}\right)$$

$$M = Ae^{\frac{1}{100}\ln\left(\frac{3}{4}\right)t}$$

When $t = 500$,

$$M = Ae^{\frac{1}{100}\ln\left(\frac{3}{4}\right)(500)}$$

$$M = Ae^{5\ln\left(\frac{3}{4}\right)}$$

$$M = 0.2373A$$

$\therefore$ 23.73% of the original substance remains after 500 years.

Example:

A radioactive substance decays at a rate that is proportional to the amount present at any time t. If 5% of the substance is decomposed after 10 hours, what percentage of the original amount is decomposed after one day?

Solution:

$$M = Ae^{kt}$$

0.05 of the substance is decomposed, then 0.95 of the substance remains.

When $t = 10$, $M = 0.95A$,

$$0.95A = Ae^{10k}$$

$$0.95 = e^{10k}$$

$$\ln(0.95) = 10k$$

$$\frac{1}{10}\ln(0.95) = k$$

$$\therefore k = \frac{1}{10}\ln(0.95)$$

$$\therefore M = Ae^{\frac{1}{10}\ln(0.95)t}$$

When $t = 1 \text{ day} = 24 \text{ hours}$,

$$M = Ae^{\frac{1}{10}\ln(0.95)(24)}$$

$$M = Ae^{\frac{12}{5}\ln(0.95)}$$

$$\therefore M = 0.8842A$$

88.42% of the original substance remains after 1 day.

$\therefore$ 11.58% of the original substance is decomposed after 1 day.

Term 2 – Week 3 – Homework

Exponential Growth and Decay:

1. Given $y = Ae^{kt}$ and $y = 20$ when $t = 0$ and $y = 25$ when $t = 2$, find A and k.

2. Given $P = Ae^{-kt}$ and $P = 2500$ when $t = 0$ and $P = 2250$ when $t = 5$, find A and k.

3. Given $Q = Ae^{kt}$ and $Q = 100$ when $t = 0$ and $Q = 150$ when $t = 10$, find A and k.

4. Given $\frac{dN}{dt} = 1.5N$ and $N = 25$ when $t = 0$, find N as a function of t.

5. Given $\frac{dM}{dt} = -0.2M$ and $M = 200$ when $t = 0$, find M as a function of t.

6. Given $\frac{dy}{dt} = 80y$, find $\frac{dy}{dt}$ when
   (i) $y = 20$.
   (ii) $t = 15$.

7. The rate of decrease of the price P of a car is given by $\frac{dP}{dt} = -kP$, where k is a positive constant and t is time in years.
   (i) If the car costs \$15000 initially and one year later it costs \$12000, find the exact value of k.
   (ii) Express P as a function of t.
   (iii) What is the rate of decrease after 2 years?
   (iv) After how many years is the price of the car halved?

8. The rate of decay of a radioactive substance is proportional to the amount of the substance available at any time t.
   (i) If half of the original amount is decomposed after 2000 years, find the yearly decay rate.
   (ii) If the original amount is 2.5 g, find the amount present after 500 years.
   (iii) What is the rate of decay after 1200 years?

9. In town A, the population size present at any time t years is given by $P = P_0e^{0.05t}$.
   (i) If in year 2000, the size of population is 2 million, find the population in year 2010.
   (ii) In which year will the population doubled?
   (iii) Find the rate of growth of population when:
      a) $P = 2.5$
      b) $t = 5$

10. Assuming the amount of bacteria present in a body follows the law of natural growth. If in one hour the number of bacteria increased from 1000 to 2500,
    (i) Show that the number of bacteria present is given by $N = 1000e^{kt}$.
    (ii) Find the value of k to the nearest 3 decimal places.
    (iii) After how many hours is the rate of growth equal to 8000 per hour?

11. A glass of boiled water is cooling and its temperature T, above the room temperature is given by $T = Ae^{-kt}$, where t is measured in minutes.
    (i) Initially the temperature of the water is 100°C, find the value of A.
    (ii) If the water cools to 80°C after 5 minutes, find
        a) The time taken to cools to 40°C.
        b) The temperature of the water after 10 minutes.
    (iii) What rate is the temperature decreasing when
        a) The temperature of the water is 60°C?
        b) After 2 minutes?

12. The rate of growth of ants in a colony is proportional to the number present at any time t. If the number of ants is doubled after 2 hours, what percentage of the original number is present after 30 minutes?

13. Two companies A and B are listed in the stock market at the same time. After one month, the share prices of company A increased from $\$6.50$ to $\$7.60$ and the share price of company B rose from $\$0.85$ to $\$2$.
    (i) Assuming the law of natural growth, show that the share prices of companies A and B in cents are given by $A = 650e^{k_1 t}$ and $B = 85e^{k_2 t}$ respectively.
    (ii) Find the values of k_1 and k_2, correct to the nearest three decimal places.
    (iii) After how many months are the share prices of the two companies equal to each other?
    (iv) What are the rates of increase of the two companies at that month?

14. A radioactive substance decays at a rate that is proportional to the amount present at any time t. If half of the substance is decomposed after 15 hours, what percentage will decompose after 10 hours?

15. The rate of decay of a radioactive substance is proportional to the mass of the substance present at any time t. If 20% decomposed after 10 years, what percentage of the original amount is decomposed after 50 years?

16. A carbon isotope decays at a rate that is proportional to its mass present at any time t. It is know that it takes approximately 4550 years for a given mass to decay by 50%. If a fossil contains 20% of the same carbon isotope, estimate the age of the fossil, giving your answer to the nearest 10 years.

End of homework