# HSC Mathematics Extension 2

## Mechanics

Term 2 – Week 2

**Student Name:** ……………………………….

**Class code:** …………………………………..

**Teacher name:** ……………………………..

DUX College

**T:** (02) 8007 6824
**E:** info@duxcollege.com.au
**W:** www.duxcollege.com.au

# Term 2 – Week 2 – Theory

**Resisted Motion in a Straight Line:**

In this section we neglect the effects of gravity and simply use Newton's Second Law to derive an equation for acceleration. We then use this equation and the initial conditions to derive equations for the velocity and displacement.

Throughout this section there will be some Resistive Force $(R)$ which is retarding the motion. This force can be proportional to $v$ , $v^2$ , $x^n$ where $n$ is real. An example of a force proportional to $v$ or $v^2$ is friction. Cases where the power of $v$ is greater than 2 are beyond this course. Using the information given we are to derive an expression for acceleration through Newton's Second Law.

We will now consider an example question to clarify this concept.

**Example:**

A particle moves along a horizontal line under a constant force P Newtons. A resistive force of magnitude mv retards the motion of the particle, where v is the velocity of the object. Find the displacement of the particle as a function of time given that at $t = 0,\ v = u\ ,\ x = 0$.

**Solution:**

Firstly we draw a diagram.


By Newton's Second Law, $\sum F = m\ddot{x} = P - mv$

$$\therefore\ \ddot{x} = \frac{P}{m} - v$$

We require the velocity as a function of time.

$$\therefore\ \frac{dv}{dt} = \frac{P}{m} - v$$

$\therefore\ \frac{dv}{P-mv} = \frac{1}{m}dt$ *(Collecting variables on either side)*

$\therefore\ \int \frac{-mdv}{P-mv} = \frac{-m}{m}\int dt$ *(Integrating with respect to the variable on either side)*

Now on the L.H.S. we have a logarithmic value since it is of the form $\int \frac{f'(x)}{f(x)}dx$ and on the R.H.S we have a linear value.

Thus, $ln(P - mv) = -t + c$

Now at $t = 0, v = u.$

Thus, $ln(P - mu) = -(0) + c$

Thus, $c = ln(P - mu)$

$$\therefore \; ln(P - mv) = -t + ln(P - mu)$$

$$ln\left|\frac{P - mv}{P - mu}\right| = -t$$

$$\therefore \; \frac{P - mv}{P - mu} = e^{-t}$$

$$\therefore \; v = \frac{P}{m} - \frac{1}{m}(P - mu)e^{-t}$$

Now we must integrate w.r.t. the resultant differential equation to obtain an expression for $x$ in terms of $t$.

$$\frac{dx}{dt} = \frac{P}{m} - \frac{1}{m}(P - mu)e^{-t}$$

$$\therefore \; x = \frac{P}{m}t + \frac{1}{m}(P - mu)e^{-t} + c_1$$

At $t = 0,\; x = 0,$

$$\therefore \; c_1 = -\frac{1}{m}(P - mu)$$

$$x = \frac{P}{m}t + \frac{1}{m}(P - mu)e^{-t} - \frac{1}{m}(P - mu)$$

$$\therefore \; x = \frac{P}{m}t + \frac{1}{m}(P - mu)(e^{-t} - 1)$$

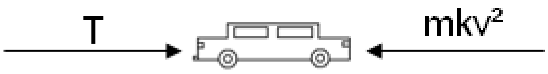
As powers of $x$ are dealt with in the Mathematics course, we shall not deal with them here. We will now consider an example question which involves the resistive force being proportional to $v^2$.

**Example:**

A car of mass $m$ kg is moving horizontally under a constant thrust of $T$ Newtons from the engine. As the car moves the net frictional force on the car is $mkv^2$, where v is the velocity of the car and $k$ is a constant of proportionality. Find an expression for $v^2$as a function of its displacement from its initial position $x$ if initially the car was stationary, and hence determine the speed of the car as $x \to \infty$.

**Solution:**

Firstly we will draw a diagram.




Now, by Newtons Second Law, $\sum F = m\ddot{x} = T - mkv^2$.

$$\therefore \quad \ddot{x} = \frac{T}{m} - kv^2$$

We require $v$ in terms of $x$ and thus we need to use the expression $v\frac{dv}{dx}$ for $\ddot{x}$.

$$v\frac{dv}{dx} = \frac{T}{m} - kv^2$$

$$\frac{v\,dv}{T - mkv^2} = \frac{1}{m}\,dx$$

On the L.H.S. we must use the Standard Integral, $\int \frac{f'(x)}{f(x)}dx = ln|f(x)| + c$

$$\frac{-2mkv\,dv}{T - mkv^2} = -2k\,dx$$

$$\therefore \quad \ln|T - mkv^2| = -2kx + c$$

At $x = 0$ , $v = 0$.

Thus, $c = ln|T|$

$$ln\left|\frac{T - mkv^2}{T}\right| = -2kx$$

$$T - mkv^2 = Te^{-2kx}$$

$$\therefore \quad v^2 = \frac{T}{mk}\left(1 - e^{-2kx}\right)$$

Now as $x \to \infty$ , $v^2 \to \frac{T}{mk}$ and thus the speed is equal to $\sqrt{\frac{T}{mk}}$.

Thus as $x \to \infty$, $speed \to \sqrt{\frac{T}{mk}}$.

**Resisted Motion of a Particle Moving Upwards Under the Influence of Gravity:**

So far we have dealt with resisted motion along a horizontal line in which there is no effect of gravity. In this section we will study the mathematical representation of an object moving upwards in a resistive medium under the influence of a gravitational force. Obviously the main difference between this and the previous section is that the force of gravity is now what was previously the thrust force. It is convention to place the direction of initial movement of the particle as positive. i.e. if the particle is moving upwards in a resistive medium, then the vertically upwards direction is positive. Also it must be noted that the questions involved will usually involve the resistive force being proportional to either $v$ or $v^2$, any power higher than the second order is beyond the Extension 2 Mathematics course except for some simple cases which are of reasonable difficulty.

One point that should be made is that the path is not symmetrical as the projectile is launched. i.e Physics students may be inclined to think that the equation used to describe the motion upwards is exactly the same as that used to describe the downwards motion. However, this is not the case, due to the direction of the gravitational force changing with respect to the positive direction assumed during the motion. The consequence of this is that a projectile moving through a resisting medium in a vertical straight line, must have its trajectory studied in two sections. Firstly one must study the upwards motion, then the downwards motion separately. This will be dealt with later on as we deal with downwards motion.

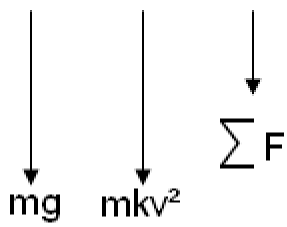
We shall now study an example question to illustrate this concept.

**Example:**

A particle with mass $m$ kg is projected vertically upwards with velocity $u\ ms^{-1}$, in a resistive medium where the resistance is proportional to the square of the velocity, under the influence of gravity. Determine the maximum height the particle reaches.

**Solution:**

Firstly we draw a diagram,




Resolving forces and using Newtons 2nd Law,

$$\sum F = m\ddot{x} = -mg - mkv^2$$

Note that we can use $mk$ instead of $k$ as the constant of proportionality due to the fact that both $m$ and $k$ are constants and thus the value of $k$ simply changes when we place $mk$ instead of $k$.

$$\therefore \ \ddot{x} = -(g + kv^2)$$

Now at maximum height the particle has zero velocity in the vertical or $x$ direction.

Thus we require the velocity $v$ in terms of the displacement $x$, to find the maximum height.

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$\frac{v\,dv}{g + kv^2} = -dx$$

$$\therefore \ \frac{2kv\,dv}{g + kv^2} = -2k\,dx$$

$$ln(g + kv^2) = -2kx + c$$

At $x = 0$ , $v = u$ , thus $c = ln(g + ku^2)$

$$\therefore \ x = \frac{1}{2k}ln\left(\frac{g + ku^2}{g + kv^2}\right)$$

At $v = 0$ , $x = \frac{1}{2k}ln\left(\frac{g+ku^2}{g}\right)$

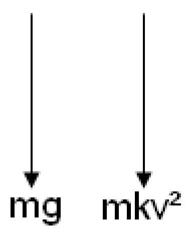
$$\therefore \ x_{max} = \frac{1}{2k}\ln\left(1 + \frac{ku^2}{g}\right)$$

**Example:**

A particle of mass $m$ kg is projected vertically upwards with a velocity of $\sqrt{\frac{g}{k}}\ ms^{-1}$. Its shape results in air resistance of magnitude of $mkv^2$. If the particle is only under the influence of the forces of air resistance and gravity, show that the time taken to reach maximum height is $\frac{\pi}{4\sqrt{gk}}$ .

**Solution:**

Firstly we draw a diagram,



By Newton's 2nd Law,

$$\sum F = m\ddot{x} = -m(g + kv^2)$$

$$\therefore\ \ \ddot{x} = -(g + kv^2)$$

We require velocity as a function of time.

$$\therefore\ \ \frac{dv}{dt} = -(g + kv^2)$$

$$\frac{dv}{g + kv^2} = -dt$$

Integrating both sides with respect to their respective variables gives,

$$\int \frac{dv}{\frac{g}{k} + v^2} = -k\ dt$$

$$\sqrt{\frac{k}{g}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) = -kt + c$$

At $t = 0$ , $v = \sqrt{\frac{g}{k}}$

$$\therefore\ \ c = \frac{\pi}{4}\sqrt{\frac{k}{g}}$$

$$\therefore\ \ \sqrt{\frac{k}{g}}\tan^{-1}\left(v\sqrt{\frac{k}{g}}\right) = -kt + \frac{\pi}{4}\sqrt{\frac{k}{g}}$$

At $v = 0$ , Max height occurs.

$$\therefore\ \ 0 = -kt + \frac{\pi}{4}\sqrt{\frac{k}{g}}$$

$$\therefore\ \ t = \frac{\pi}{4\sqrt{gk}}$$

**Resisted Motion of a Particle Moving Downwards Under the Influence of Gravity**

We shall now study the motion of a particle moving downwards under the influence of gravity in a resistive medium. The only difference now is that we have the force due to gravity being positive with it being in the direction of motion and the resistive force being negative, in the opposing direction of motion to the particle.

In this section the important concept of *terminal velocity* arises. The terminal velocity of a particle is the velocity the particle reaches as the Net acceleration of the particle tends to zero. It can be found by setting $\ddot{x}$ to zero in the acceleration equation. It can also be found by finding the limiting value of $v$ as either $t$ or $x$ tends to infinity.

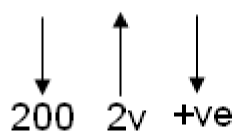
In exam questions, it is usual to combine both upwards motion and downwards motion in one question. Thus students must understand that the equation of motion for downwards motion is different to that of upwards in the same medium and that thus the resultant motion is in fact not symmetrical. Thus the downwards motion must be treated separately to the upwards motion. We shall deal with questions of this nature in the examples ahead.

**Example:**

A particle of mass 20 kg is dropped in a medium with resistance $2v$ Newtons where $v$ is the speed of the particle. Assume the acceleration due to gravity is 10 $m/s$. Find a value for the terminal velocity, and hence find the distance fallen by the particle when it has reached half the terminal velocity.

**Solution:**

Firstly we draw a diagram,




By Newton's 2nd Law,

$$\sum F = 20\ddot{x} = 200 - 2v$$

$$\ddot{x} = \frac{1}{10}(100 - v)$$

Now as $\ddot{x} \to 0,\ v \to 100$

Thus the terminal velocity is *100* $m/s$

Now we need to solve the differential equation to obtain an equation in terms of $v$ and $x$.

$$v\frac{dv}{dx} = \frac{1}{10}(100 - v)$$

$$\frac{v\,dv}{100 - v} = \frac{1}{10}dx$$

$$\therefore \left(1 - \frac{100}{100 - v}\right)dv = -\frac{1}{10}dx \qquad \textit{(Rational function)}$$

$$v + 100\,ln|100 - v| = -\frac{1}{10}x + c$$

At $x = 0$, $v = 0$, $\therefore\ c = 100\ln 100$

$$\therefore\ x = 1000\,ln\left|\frac{100}{100 - v}\right| - 10v$$

Now, at $v = \frac{1}{2} \times 100 = 50\,ms^{-1}$, $x = 1000\ln 2 - 500$

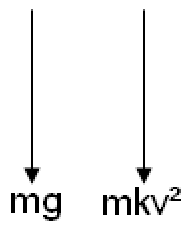
Thus, the distance travelled when the velocity is half the terminal velocity is $500(2\ln 2 - 1)$ metres.

**Example:**

A particle is projected vertically upwards under the influence of gravity, in a medium where the resistance to the particle's motion has a magnitude of $mkv^2$, where m is the mass of the particle, $v$ is the velocity of the particle, and $k$ is a constant of proportionality. If the particle is projected with velocity $\sqrt{\frac{g}{k}}$ metres per second, show that the time it takes for the particle to return to its initial position is $\frac{1}{\sqrt{gk}}\left[ln\left(\sqrt{2} + 1\right) + \frac{\pi}{4}\right]$ seconds.

**Solution:**

Firstly we must consider the upwards motion to find the time taken to reach maximum height.




$$\sum F = m\ddot{x} = -m(g + kv^2)$$

$$\ddot{x} = -(g + kv^2)$$

$$\therefore\ t_{up} = \frac{\pi}{4\sqrt{gk}}$$

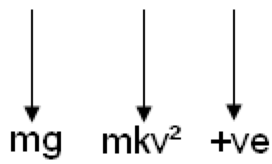
Now, we must find the maximum height reached to find the time the particle has to reach the point of projection from max height.

Now, to find the maximum height ($h$) reached we will use the result from example 3. The only difference in this case is that $u = \sqrt{\frac{g}{k}}$.

From example 3, $h = \frac{1}{2k} ln\left(1 + \frac{k\left(\sqrt{\frac{g}{k}}\right)^2}{g}\right)$

$\therefore \quad h = \frac{1}{2k} ln\, 2 \; Metres$

Now for the downwards motion, we derive the equation of motion from Newton's 2nd Law.




$$\sum F = m\ddot{x} = mg - mkv^2$$

$$\therefore \quad \ddot{x} = g - kv^2$$

Now we require the equation in terms of $v$ and $x$ to find the value of $v$ at $x = h$, i.e. when the particle drops from the maximum height the distance travelled from the point of projection to maximum height is the distance fallen in the downwards motion.

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{v\, dv}{g - kv^2} = dx$$

$$\frac{-2kv\, dv}{g - kv^2} = -2k\, dx$$

$\therefore \quad ln|g - kv^2| = -2kx + c$

At $x = 0, v = 0 \therefore c = ln\, g$

Thus, at $x = h = \frac{1}{2k} ln\, 2,\; ln|g - kv^2| = ln\left(\frac{g}{2}\right)$

$$\therefore \quad v = \sqrt{\frac{g}{2k}}$$

Now we must find $v$ in terms of $t$.

$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dv}{g - kv^2} = dt$$

Now on the L.H.S we could use partial fractions, but an easier method is through the use of the standard integral,

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} ln\left|\frac{a + x}{a - x}\right| + c$$

$$\therefore \quad \frac{dv}{\frac{g}{k} - v^2} = k\, dt$$

$$\frac{\sqrt{k}}{2\sqrt{g}} ln\left|\frac{\sqrt{\frac{g}{k}} + v}{\sqrt{\frac{g}{k}} - v}\right| = kt + c_1$$

At $t = 0,\ v = 0\,, \therefore\ c_1 = 0$

Now at $v = \sqrt{\frac{g}{2k}},\ t = \frac{1}{2\sqrt{gk}} ln\left(\frac{\sqrt{\frac{g}{k}} + \sqrt{\frac{g}{2k}}}{\sqrt{\frac{g}{k}} - \sqrt{\frac{g}{2k}}}\right)$

i.e. $t = \frac{1}{2\sqrt{gk}} ln\left(\frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}\right)$

$$t = \frac{1}{2\sqrt{gk}} \ln\left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}\right)$$

$$\therefore\ t_{down} = \frac{1}{\sqrt{gk}} \ln\left(\sqrt{2} + 1\right)$$

$$\therefore\ t_{total} = t_{up} + t_{down}$$

$$= \frac{\pi}{4\sqrt{gk}} + \frac{1}{\sqrt{gk}} \ln\left(\sqrt{2} + 1\right)$$

$\therefore\ t_{total} = \frac{1}{\sqrt{gk}}\left[\frac{\pi}{4} + ln\left(\sqrt{2} + 1\right)\right]$ *Seconds.*

# Term 2 – Week 2 – Homework

**Resisted Motion in a Straight Line:**

1. A particle of mass $m$ kg is projected from the origin with velocity $U$ under the influence of gravity, at an angle $\theta$ along an $2D$ plane in a medium which exerts a resistance force proportional to the velocity of the particle. Determine the limiting value for $x$ if the origin, is positioned very high up in the physical, and also show that $\dot{y} = U\sin\theta\, e^{-kt} + \frac{g}{k}\left(e^{-kt} - 1\right)$.

2. A mass of $m$ kg experiences a constant force of magnitude $R$ Newtons. The frictional force on the object has a magnitude $mkv$ Newtons, where $k$ is a positive constant and $v$ is the speed of the particle, measured in m /s. Express $v$ in terms of $t$ and hence determine the limiting velocity of the particle, given that initially, the particle was moving at $U$ m /s.

3. A car of mass 2 tonnes initially at rest, has a constant thrust of 250 $T$ Newtons applied onto each wheel. If each wheel experiences a resistive frictional force of $125v$ Newtons where $v$ is the velocity in m /s, determine the maximum velocity that the car can travel at, given that 250 $T$ Newtons is the maximum output of the engine.

4. A submarine of mass $m$ is travelling underwater at maximum power. At maximum power, its engines deliver a force $F$ on the submarine. The water exerts a resistive force proportional to the square of the submarine's speed $v$. The submarine increases its speed from $v_1$ to $v_2$. Show that the distance travelled during this period is $\frac{m}{2k} ln\left[\frac{F-kv_1^2}{F-kv_2^2}\right]$.

5. A particle of mass 1 kg is projected from a point with velocity $u$ m /s along a smooth horizontal table in a medium whose resistance is $Rv^2$ Newtons when the particle has any velocity $v$ m /s, $R$ being constant. Find the reciprocal of its velocity after $t$ seconds.

6. An identical particle to the one in the above question is projected simultaneously from $O$ with the first particle but vertically upwards under gravity with velocity $u$ in the same medium. Show that the velocity $V$ of the first particle when the second is momentarily at rest is given by $\frac{1}{V} = \frac{1}{u} + \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$, where $Ra^2 = g$.

7. A body of mass $m$ is projected vertically upwards from the ground with speed $u_0$. The force due to gravity acting on the body is constant but there is a resisting force of magnitude $mkv^2$ at speed $v$. Show that the maximum height $H$ which the body reaches is given by $2kH = ln\left(\frac{g+ku_0^2}{g}\right)$ and that the speed $v_0$ with which the body reaches the ground is given by $2kH = ln\left(\frac{g}{g-kv_0^2}\right)$. Using these results show that $\frac{1}{v_0^2} = \frac{1}{u_0^2} + \frac{k}{g}$.

8. An object of mass 20 kg is dropped through the atmosphere, meeting an air resistance of $2v$ Newtons at speed $v\ ms^{-1}$. Acceleration due to gravity is 10 m /s².
   a) Find the expression for velocity $v$ at time $t$ seconds.
   b) Show that the distance $x$ m travelled with velocity is given by $x = 1000\,ln\left(\frac{100}{100-v}\right) - 10v$.
   c) Use parts (a) and (b) to calculate the distance the object falls in the first 10 seconds.

9. A particle is thrown vertically upwards with initial speed $V_0$ in a medium for which the resistance is proportional to the particle's speed. The particles terminal speed is $V_T$. Show that,
   a) The time interval during which the particle is moving upwards is $\frac{V_T}{g}\,ln\left(1+\frac{V_0}{V_T}\right)$.
   b) The maximum height that the particle attains is $\frac{V_T V_0}{g} - \frac{{V_T}^2}{g}\,ln\left(1+\frac{V_0}{V_T}\right)$.
   c) If the particle returns to the point of projection with speed $V$ after a time $T$ from the instant of projection, show that $gT = V_0 + V$.

10. A train of mass $m$, pulled by a locomotive which exerts a constant (propelling) force $P$ is moving at speed $v$ along a straight level track against a resistive force $mkv$, where $k$ is a positive constant. Show that if the speed increases from 2 m /s to 4m /s over a time interval of 5 seconds.
    a) $P = 2km\left(\frac{2e^{5k}-1}{e^{5k}-1}\right)$
    b) Find the corresponding distance moved.
    c) Prove that there is an upper bound to the speed that the train can attain and find the value of this upper bound.

11. A passenger liner has a mass $M$. When the liner has speed $v$ the water resistance to its motion has magnitude $Mv^2$.
    a) The liner moves from rest under a constant force $Mf$ produced by the propellers. If $x$ is the distance travelled by the liner at time $t$ after the propellers have been started, show that $\ddot{x} = f - v^2$. Hence show that after travelling a distance $d_1$ the speed $v_1$ of the liner s given by ${v_1}^2 = f\left(1-e^{-2d_1}\right)$.
    b) At the instant the liner has travelled a distance $d_1$ there is an emergency and the propellers are reversed. The liner comes to rest under the retarding force $Mf$ produced by the propellers after travelling a further distance $d_2$. If $x$ is the distance travelled by the liner at time $t$ after the propellers have been reversed, show that $\ddot{x} = -(f+v^2)$. Hence show that $e^{2d_2} + e^{-2d_1} = 2$.

12. A particle is moving in the positive direction along a straight line in a medium that exerts a resistance to motion proportional to the cube of the velocity. No other forces act on the particle, that is, $\ddot{x} = -kv^3$, where $k$ is a positive constant. At time $t = 0$, the particle is at the origin and has velocity $U$. At time $t = T$, the particle is at $x = D$ and has velocity $V$.
    a) Using the identity $\ddot{x} = \frac{dv}{dt}$ show that $\frac{1}{V^2} - \frac{1}{U^2} = 2kT$.
    b) Using the identity $\ddot{x} = v\frac{dv}{dx}$, show that $\frac{1}{V} - \frac{1}{U} = kD$.
    c) Hence show that $\frac{D}{T} = \frac{2UV}{U+V}$.

13. An object of mass $m$ is set in motion with speed $20\ ms^{-1}$ and moves in a straight line. Subsequently the only force acting on the object directly opposes its motion and has magnitude $4m + 0.01mv^2$, where $v\ ms^{-1}$ is the speed at time $t$ seconds.
    a) Show that the acceleration $a$ m /s² of the object at time $t$ seconds is given by $a = -4 - 0.01v^2$.
    b) Find the time taken by the object before it comes to rest.
    c) Find the distance travelled by the object before it comes to rest.

14. A particle of mass $m$ is projected vertically upwards under gravity. The air resistance to the motion is $\frac{1}{100}mgv^2$ where $v$ is the speed of the particle.
    a) Show that during the upward motion of the particle, if $x$ is the upward vertical displacement of the particle from its projection point at time $t$, then $\ddot{x} = -\frac{1}{100}g(100 + v^2)$. If the speed of projection is $u$, show that the greatest height (above the projection point) reached by the particle is $\frac{50}{g}ln\left(\frac{100+u^2}{100}\right)$.
    b) Show that during the downward motion of the particle, if $x$ is the downward vertical displacement of the particle from its highest position at time $t$ after it begins the downward motion, then $\ddot{x} = \frac{1}{100}g(100 - v^2)$. Show that the speed of the particle on return to its point of projection is $\frac{10u}{\sqrt{100+u^2}}$.
    c) Find the terminal velocity $V_0$ of the particle for the downward motion. If the initial speed of projection of the particle is $V_0$, show that the speed on return to the point of projection is $\frac{1}{\sqrt{2}}V_0$.

15. A particle of mass $m$ kg is dropped from rest in a medium where the resistance to the motion has a magnitude $mkv$ where the particle has a velocity $v$ m /s.
    a) Show that the equation of motion of the particle is $\ddot{x} = k(V_0 - v)$ where $V_0$ m /s is the terminal velocity of the particle in this medium, and $x$ metres is the distance fallen after $t$ seconds.
    b) Find in terms of $V_0$ and $k$ the time $T$ seconds taken for the particle to attain $50\%$ of its terminal velocity, and the distance fallen in this time.
    c) What percentage of its terminal velocity will the particle have attained in time $2T$ seconds? Sketch a graph of $v$ against $t$ showing this information.
    d) If the particle has reached $87.5\%$ of its terminal velocity in 15 seconds, find the value of $k$.

16. A toy of mass $m$ kg has a parachute device attached. It is released from rest at the top of a vertical cliff 40 m high. During its fall, the forces acting are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{10}mv^2$ when the speed of the toy is $v$ m /s. After $2\ln 2$ seconds, the parachute disintegrates, and then the only force acting on the toy is gravity. The acceleration due to gravity is taken as $g = 10$ m /s². At time $t$ seconds, the toy has fallen a distance $x$ metres from the top of the cliff, and its speed is $v$ m /s².
    a) Show that while the parachute is operating, $10\ddot{x} = 100 - v^2$. Hence show that $v = 10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$ and $x = -5\,ln\left[1 - \left(\frac{v}{10}\right)^2\right]$.
    b) Find the exact speed of the toy and the exact distance fallen just before the parachute disintegrates.
    c) After the parachute disintegrates, find an expression for $\ddot{x}$ and use integration to find the speed of the toy just before it reaches the base of the cliff. Give your answer correct to 2 significant figures.

17. A car of mass $m$ kg running without power on a straight road, experiences a resistive force equal to $m(v + v^3)$ where $v$ is the velocity of the car. The car initially had a speed of $U$ m /s.
    a) Show that the distance $x$, the car has travelled after slowing down to a speed $v_0$ is $x = tan^{-1}\left(\frac{U-v_0}{1+Uv_0}\right)$.
    b) Show that the time needed to slow down to this speed is $t = \frac{1}{2}ln\left(\frac{U^2(1+{v_0}^2)}{{v_0}^2(1+U^2)}\right)$.

18. A particle of mass $m$ kg is launched vertically upwards in a resistive medium at a velocity 5 m /s. It is subject to the force of gravity and to a resistance of magnitude $\frac{mv^3}{100}$, where $v$ is the velocity of the particle. Take $g = 10$ m /s² and upwards as positive.
    a) Find the equation of motion, and thus show that the height $x$ above the point of launch and the velocity $v$ are related by $\frac{dv}{dx} = \frac{v^3+1000}{-100v}$.
    b) Find constants $a, b$ and $c$ such that,
    $$\frac{100v}{v^3 + 1000} = \frac{a}{v + 10} + \frac{bv + c}{v^2 - 10v + 100}$$
    c) Hence find the maximum height reached by the particle, giving your answer correct to the nearest centimetre.

**End of homework**